CS109 April 28, 2016

Practice Problems for Midterm Review

1. **James has 8 friends, but can only invite 5 to a party. How many choices does he have if 2 are the friends are feuding and don't want to go together?**

There are a total of $\binom{8}{5}$ choices. Of these, there are $\binom{2}{2}\binom{6}{3}$ that include the two people at the party.

$$\binom{8}{5} - \binom{2}{2}\binom{6}{3} = 36$$

Another way to look at it is that there are three cases. In the first case, only the first friend goes, which is $\binom{6}{4}$ (you only get to pick combinations of the 4 other guests). The second case is that only the second friend goes, which is $\binom{6}{4}$ as well. The third case is that neither of them goes, which is $\binom{6}{5}$.

$$\binom{6}{4} + \binom{6}{4} + \binom{6}{5} = 36$$

2. **We find that for a certain CS assignment, 36 percent of submissions are in C and 22 percent of assignments coded in C have bugs. We are also told that 30 percent of the submissions has bugs.**

   a. **What is the probability that a randomly selected submission is both in C and has bugs?**

   We know that $P(B) = 0.3, P(C) = 0.36, P(B|C) = 0.22$. Thus:

$$P(BC) = P(B|C)P(C) = 0.22 \times 0.36 = 0.0792$$

   b. **What is the conditional probability that a randomly selected submission is in C given that it has a bug?**

$$P(C|B) = \frac{P(B|C)P(C)}{P(B)} = \frac{(0.22 \times 0.36)}{0.3} = 0.264$$

3. **Prove that if $P(A|B) > P(A)$ then $P(B|A) > P(B)$.**

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} > P(A)$$

Since $P(A)$ and $P(B)$ are both positive:

$$\frac{P(B|A)P(A)}{P(B)} > P(B) \rightarrow \frac{P(B|A)P(A)}{P(A)} > P(B) \rightarrow P(B|A) > P(B)$$

4. **If X is a binomial random variable where $E[X] = 6$ and $Var(X) = 2.4$, what is $P(X = 5)$?**

$$E[X] = np = 6$$

$$Var(X) = n(1-p) = 2.4$$

Solving the system of two equations, we find that $n = 10$ and that $p = 0.6$. Therefore:

$$P(X = 5) = \binom{10}{5}(0.6)^5(0.4)^5$$

5. **A company makes USB cables. The probability that a cable is faulty is 0.01. What is the probability that a sample of 1000 will have exactly 8 defective cables?**

   a. **Exact?**

$$P(X = 8) = \binom{1000}{8}(0.01)^8(0.99)^{1000-8} = 0.112834$$

   b. **Approximate?**

$$\lambda = np = 0.01 \times 1000 = 10$$

$$P(X = 8) = \frac{e^{-10}10^8}{8!} = 0.11260$$

6. **The life of a specific computer is distributed with mean 1000 days and standard deviation 400 days.**

   a. **What is the probability that the computer lasts more than 1200 days?**

$$Z = \frac{(1200\text{-}1000)}{400} = 0.5$$

$$P(X \leq 1200) = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085$$

   b. **Given that the computer already lasted 1000 days, what's the probability that it lasts at least 500 more days?**

$$P(X > 1500 \mid X > 1000) = \frac{P(X > 1500)}{P(X > 1000)} = \frac{P\left(Z > \frac{1500 - 1000}{400}\right)}{P\left(Z > \frac{1000 - 1000}{400}\right)}$$

$$= \frac{P(Z > 1.25)}{P(Z > 0)} = \frac{1 - 0.8943}{0.5} = 0.2114$$

7. **There are 7184 undergrads at Stanford. Each student has a 13% probability of taking CS106A at some point in their four years here. Let X be the number of students who take CS106A at Stanford. What is the approximate probability that more than 900 of these students will take CS106A?**

Since n is large, we use the normal approximation to the binomial.

We know that $X \sim Bin(7184, 0.13)$.

Therefore, $E[X] = np = 933.92$ and $Var(X) = np(1-p) = 812.51$

We therefore approximate X as $Y \sim N(933.92, 812.51)$:

$$P(X > 700) \approx P(Y > 700.5) = P\left(Z > \frac{900.5 - 933.92}{\sqrt{812.51}}\right) = P(Z > -1.17) = P(Z < 1.17) = 0.879$$

8. **The joint density function of X and Y is:**

$$f_{X,Y}(x, y) = \begin{cases} xy & 0 < x < 1, \ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

   a. **Find the marginal probability density function of X.**

$$f_X(x) = \int_0^2 xy \, dy = x \int_0^2 y \, dy = 2x$$

   b. **Find $E[X]$.**

$$E[X] = \int_0^\infty x f_X(x) \, dx = 2 \int_0^1 x^2 \, dx = \frac{2x^3}{3}$$

   c. **What's the conditional PDF $f_{Y|X}(y|x)$**

   By definition:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

   Therefore:

$$f_{Y|X}(y|x) = \frac{xy}{2x} = \frac{y}{2}$$

d. Are X and Y independent?

Yes:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \text{ where } f_X(x) = 2x \text{ and } f_Y(y) = \frac{y}{2}$$

9. We have an array of Boolean variables, where n of them are 0 and m of them are 1. We remove them from the bag in a randomly chosen order. What is the expected number of instances in which a 0 is immediately followed by a 1?

Let us define an indicator variable I_j as follows:

$$I_j = \begin{cases} 1 & \text{the } j^{th} \text{ variable is 0 and the } (j+1)^{th} \text{ variable is 1} \\ 0 & \text{otherwise} \end{cases}$$

Thus, we are trying to find:

$$X = \sum_{j=1}^{n+m-1} I_j$$

We know that the expected value of the sum is the sum of the expected value. Therefore:

$$E[X] = E\left[\sum_{j=1}^{n+m-1} I_j\right] = \sum_{j=1}^{n+m-1} E[I_j]$$

For an indicator variable, the expected value is p. Therefore:

$$\begin{aligned} E[I_j] &= P(\text{the } j^{th} \text{ variable is 0 and the } (j+1)^{th} \text{ variable is 1}) \\ &= P(\text{the } (j+1)^{th} \text{ variable is 1} \mid \text{the } j^{th} \text{ variable is 0})P(\text{the } j^{th} \text{ variable is 0}) \end{aligned}$$

$$P(\text{the } j^{th} \text{ variable is 0}) = \frac{n}{n+m}$$

$$P\left(\text{the } (j+1)^{th} \text{ variable is } 1\right) = \frac{m}{n+m-1}$$

Therefore:

$$E[X] = \sum_{j=1}^{n+m-1} \frac{n}{n+m} \frac{m}{n+m-1} = \frac{nm}{n+m}$$